Exponential, logarithmic and nonlinear functions

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Definition 1 gives a definition of a function. Example 1 gives some examples of this. In Definition 2 we talk about variables and parameters of a function.

Definition 1. A *function* is an operator or a procedure which accepts a permissible input and transforms it into a unique output. The input is some nonempty set. If a function is defined to be $y = \mathrm{f}(\mathrm{x})$, then x is the input vector, y the output, and $\mathrm{f}(\cdot)$ the function itself.

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Example 1. If $\mathrm{f}(\cdot)$ is the function of dressing, then its input is possibly a person and its output a dressed person. If $\mathrm{f}(\cdot)$ is the function of making up, then the input is perhaps a girl and the output a made-up girl.

Definition 2. Let $y = \mathrm{f}(\mathrm{a}, \mathrm{x})$ be a function, where a is a set of all its parameters, and x a set of all its variables. Then y is its dependent variable and x_i, for all $i \in \mathrm{x}$, are its independent variables. In other words, x_i vary, y follows, and a_i could assume any value within the range of its permissible ones, but its value must be constant.

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Definition 3 and Example 2 address inverse function, and respectively operator and inverse operator.

Definition 3. An *inverse function* is an expression of the independent variable in terms of the dependent variables. The inverse of the function $\mathrm{f}(\cdot)$ is denoted by $\mathrm{f}^{-1}(\cdot)$. If $\mathrm{f}(\cdot)$ is a function which admits one independent variable, namely x, then one could express it as,

$$y = \mathrm{f}(x) \tag{1}$$

Its inverse function is then,

$$\mathrm{f}^{-1}(y) = x \tag{2}$$

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Example 2. Both the function and its inverse may be thought of as being an operator operating on an input to produce an output. The function,

$$y = \mathrm{f}(x)$$

is understood diagrammatically as,

$$y \leftarrow \boxed{\ \mathrm{f}(\cdot)\ } \leftarrow x$$

while its inverse function,

$$x = \mathrm{f}^{-1}(y)$$

is displayed as a diagram as,

$$y \rightarrow \boxed{\ \mathrm{f}^{-1}(\cdot)\ } \rightarrow x$$

Theorem 1 is related to the domain and range of inverse functions. Example 3 gives some examples of inverse functions.

Theorem 1. An inverse function must always be a one-to-one mapping.

Proof. Let $\mathrm{f}(\cdot)$ be a function. Then $\mathrm{f}(\cdot)$ can be either one-to-one or many-to-one, and therefore $\mathrm{f}^{-1}(\cdot)$ could turn out to be either one-to-one or one-to-many. But since f^{-1} is also a function, so for each of the values in its domain the corresponding value in its range must be unique. This means that in cases where f^{-1} turns out to be one-to-many, some constraints must be put on its input in order to make the output one-to-one, which then makes all the outputs from $\mathrm{f}^{-1}(\cdot)$ one-to-one. ¶

Example 3. Table 1 gives some of the functions and their corresponding inverse functions which are fundamental in mathematics.

$\mathrm{f}(\cdot)$	$\mathrm{f}^{-1}(\cdot)$
addition	subtraction
multiplication	division
power	root
exponential	logarithm

Table 1 *Some of the functions and their corresponding inverse functions.*

Then, letting a be a constant, Table 1 becomes Table 2

$\mathrm{f}(\cdot)$	$\mathrm{f}^{-1}(\cdot)$
$x+a$	$x-a$
$x \cdot a$	$\frac{x}{a}$
x^a	$\sqrt[a]{x}$
a^x	$\log_a x$

Table 2 *The notational forms of functions and their inverses.*

in which division and logarithm are both undefined for $a = 0$.

Definition 4 gives some of the basic building blocks of mathematics. Example 4 then shows how these are built one on top of another.

Definition 4. The inverse of the addition,

$$y = x + a$$

is the subtraction,

$$y - a = x$$

The inverse of the multiplication,

$$y = ax$$

is the division,

$$\frac{y}{a} = x$$

The inverse of the power,

$$y = x^a$$

is the root,

$$\sqrt[a]{y} = x$$

The inverse of the exponential,

$$y = a^x$$

is the logarithm,

$$\log_a y = x$$

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Example 4. Figure 1 is a diagram which shows how addition makes multiplication makes power function.

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\begin{array}{ccccc}
 & & \underbrace{x \cdots x}_{b} & \rightarrow & x^b \\
 & & \uparrow & & \\
 & & x \cdot x = x^2 & & \\
 & & \uparrow & & \\
\underbrace{x + \ldots + x}_{a} & \rightarrow & x \cdot a & & \\
\uparrow & & & & \\
x + x = 2x & & & &
\end{array}
$$

Figure 1 *The building blocks of mathematics, from addition to multiplication to power function.*

In Figure 1 we start from considering a variable at the base. If instead of doing this we begin by considering addition of some constant a, then eventually a becomes a parameter in our more complicated functions.

$$\begin{array}{ccccc} & & \underbrace{a \cdots a}_{x} & \rightarrow & a^x \\ & & \uparrow & & \\ \underbrace{a + \ldots + a}_{n} & \rightarrow & a \cdot n & & \\ \uparrow & & & & \\ a + a = 2a & & & & \end{array}$$

Figure 2 *Starting from a constant to obtain in the end the exponential function.*

Our derivation in Figure 1 gives us x^b when b is an integer, and similarly that in Figure 2 gives a^x when x is an integer, but both the power- and the exponential functions can be extended to cover cases where the powers are noninteger, that is to say, when they are real or complex numbers. In these cases, however, the output may no longer be real.

Next, we look at the exponential function, which is defined in Definition 5. Example 5 discusses this further, and the exponential to the power of zero is looked at in Theorem 2.

Definition 5. An *exponential function* is defined as $y = a^x$, where $a > 0$ and $a \neq 1$.

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Example 5. The domain of the exponential function $y = a^x$ is the set of all real numbers, while its range the set of all positive real numbers. The function is convex and increasing when $a > 1$, and convex and decreasing when $0 < a < 1$. At $x = 0$, the value of the function is $y = 1$ for any $a > 0$.

Theorem 2. For any $a \neq 0$,

$$\lim_{x \to 0} a^x = 1$$

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Problem 1. Try prove Theorem 2.

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Theorem 3 gives some of the rules of exponential function and Example 5 looks at some exponential function.

Theorem 3. Three basic rules of the exponential function are,

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

Proof. Write, say, a^m as,

$$\underbrace{a \cdots a}_{m}$$

and similarly for a^n. Then all three equations above become obvious. ¶

Example 6. Figure 3 gives a graph of the exponential function when $a > 1$. Figure 4 gives a graph of the exponential function $y = a^x$ when $0 < a < 1$.

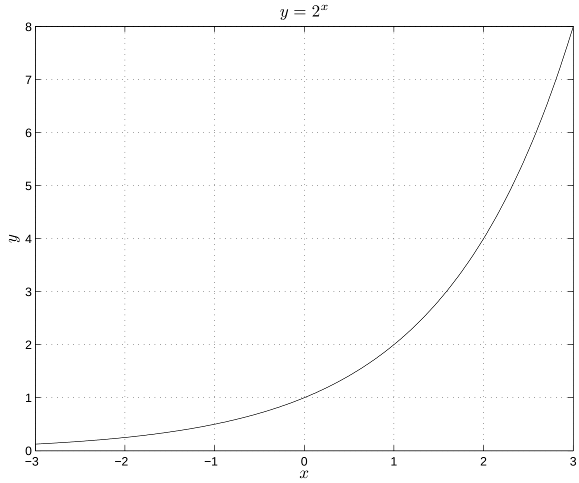

Figure 3 *Example of the graph of the exponential function when $a > 1$. Here the graph is that of $y = 2^x$.*

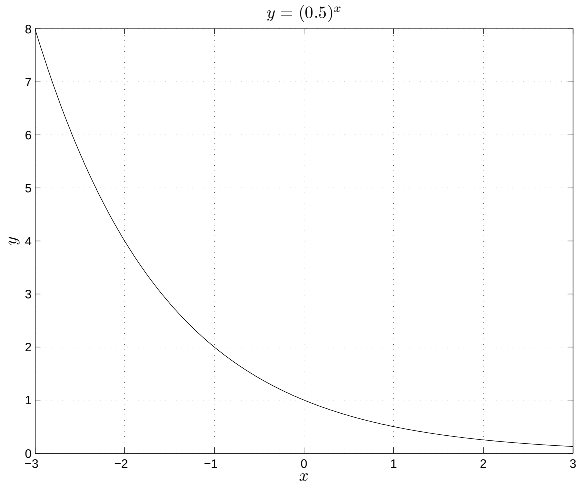

Figure 4 *An example of graph of the exponential function $y = a^x$ when $0 < a < 1$. Here* $a = 0.5$.

From Figure's 3 and 4, one may see that the graph of $y = a^x$, where $0 < a < 1$, is the same as the graph of $y = b^{-x}$, where $\frac{1}{a} = b > 1$. This is obvious since by putting $a = \frac{1}{b}$ into $y = a^x$ one arrives at $y = b^{-x}$, and if $0 < a < 1$, then $b > 1$.

One of the place we find use of an exponential function is the growth- and decay curves mentioned in Definition 6. Example 7 gives several basic growth functions.

Definition 6. Let $a > 1$. Then the graph of $y = a^x$ is called a *growth curve*, while that of $y = a^{-x}$ is called a *decay curve*.

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Example 7. There are basically three laws of growth, namely unlimited, limited and logistic growth, all of which involve an exponential function. The model is for unlimited growth,

$$y(t) = ae^{rt}$$

for limited growth,

$$y(t) = m\left(1 - e^{-rt}\right)$$

and for logistic growth,

$$y(t) = \frac{m}{1 + ae^{-rmt}}$$

where a, m and r are constants.

Interests, and future- and present values are discussed in Example's 8 and 9.

Example 8. The value of a principal p compounded annually at an interest rate i for t years is,

$$s = p(1+i)^t$$

where i is expressed in decimal points. For compounding m times a year, then,

$$s = p\left(1 + \frac{i}{m}\right)^{mt}$$

If the compounding is continuous, at 100 per cent interest for one year, then,

$$s = p \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m} = pe$$

where e is the Euler's constant, $e = 2.71828\ldots$

Example 9. For multiple compounding,

$$p(1+i_e)^t = p\left(1 + \frac{i}{m}\right)^{mt}$$

the effective annual rate of interest is,

$$i_e = \left(1 + \frac{i}{m}\right)^{m} - 1$$

The effective annual rate of interest for continuous compounding is,

$$i_e = e^r - 1$$

Definition 7 and Example 10 are about discounting.

Definition 7. *Discounting* is the process of finding the present value p of a future sum of money s.

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Example 10. Discounting when under annual compounding is,

$$s = p(1+i)^t$$

when under multiple compounding,

$$p = s\left(1 + \frac{i}{m}\right)^{-mt}$$

and when under continuous compounding,

$$p = se^{-rt}$$

When discounting, the interest rate i is called the *rate of discount*.

Example 11. A discrete growth $s = p(1+i/m)^{mt}$ can be converted to a continuous growth $s = pe^{rt}$ thus,

$$p\left(1 + \frac{i}{m}\right)^{mt} = pe^{rt}$$

$$\ln\left(1 + \frac{i}{m}\right)^{mt} = \ln e^{rt}$$

$$r = m\ln\left(1 + \frac{i}{m}\right)$$

Therefore,

$$s = p\left(1 + \frac{i}{m}\right)^{mt} = pe^{m\ln\left(1+\frac{i}{m}\right)t}$$

Example 12. Reversing the sign of x, that is replacing x by $-x$, has the effect of reflection of the original graph with respect to the y-axis. Reversing the sign of y, that is replacing y by $-y$, gives a reflection of the same with respect to the x-axis. The graphs of $y = a^{\pm x}$ remain always above the x-axis, in other words the function $y = a^{\pm x}$ maps $-\infty < x < \infty$ to $y > 0$. The two functions $y = a^x$ and $y = a^{-x}$ are the reflection of each other with respect to the y-axis. It can be easily seen that the functions $y = -a^{\pm x}$ are the reflection with respect to the x-axis respectively of $y = a^{\pm x}$.

Definition 8 introduces the logarithmic function, and Example 13 gives some elaboration regarding this. Some examples of natural logarithm are given in Example 14. Theorem 4 gives rules for logarithm.

Definition 8. The *logarithmic function* with base a is defined to be the inverse of the exponential function, and is written $y = \log_a x$, where $a > 0$ and $a \neq 1$. The logarithmic function of base 10 is called the *common logarithmic function*, and one of base e, where $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is called the *natural logarithmic function*. By the notation $y = \log_a x$ we mean that the logarithm base a of x is the power to which a must be raised to get x.

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Example 13. The domain of the logarithmic function $y = \log_a x$ is the set of all positive real numbers, its range the set of all real numbers. The function is concave and increasing for $a > 1$, and is convex and decreasing for $0 < a < 1$. Note also that $\log_a x$ is the power which a must be raised to get x.

Example 14. Note that $e^{\ln a} = a = \ln e^a$ where $a > 0$, $e^{\ln x} = x = \ln e^x$ where $x > 0$, and $e^{\ln \mathrm{f}(x)} = \mathrm{f}(x) = \ln e^{\mathrm{f}(x)}$ where $\mathrm{f}(x) > 0$.

Theorem 4. Four basic rules for logarithm function are listed in the following.

$$\log_b m + \log_b n = \log_b mn$$

$$\log_b m - \log_b n = \log_b \frac{m}{n}$$

$$\log_b m^z = z \log_b m$$

$$\log_b n = \frac{\log_x n}{\log_x b}$$

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Problem 2. Prove Theorem 4, the theorem for rules of logarithm.

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Definition 9 sets out the meaning of the elasticity of substitution. Example 15 discuss the values of the elasticity of substitution. Definition 10 is on the constant elasticity of substitution production function.

Definition 9. The *elasticity of substitution* σ is defined as,

$$\sigma = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\frac{k}{l}}}{\frac{\mathrm{d}\left(\frac{p_l}{p_k}\right)}{\frac{p_l}{p_k}}} = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\mathrm{d}\left(\frac{p_l}{p_k}\right)}}{\frac{\frac{k}{l}}{\frac{p_l}{p_k}}}$$

where $\frac{k}{l}$ is called the *least-cost input ratio*, and $\frac{p_l}{p_k}$ the *input-price ratio.*

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Example 15. The value $\sigma = 0$ means there is no substitutability, that is the two inputs are complements of each other and both must be used together in a fixed proportion. The value $\sigma = \infty$ means that the two goods may substitute each other perfectly. Ultimately, $0 \leq \sigma \leq \infty$.

Definition 10. A *constant elasticity of substitution production function* is a production function where, unlike the Cobb-Douglas function, has an elasticity of substitution whose value is constant but not necessarily 1. In its typical form, it is,

$$q = a\left(\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right)^{-\frac{1}{\beta}}$$

where a is called the *efficiency parameter*, α the *distribution parameter*, β the *substitution parameter*. Furthermore, β determines σ, and $a > 0$, $0 < \alpha < 1$, and $\beta > -1$.

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Example 16 discusses logarithmic transformation of nonlinear functions.

Example 16. Some nonlinear functions can be converted to linear functions using logarithmic transformation, for example the Cobb-Douglas production function,

$$q = ak^{\alpha}l^{\beta}$$

which becomes

$$\ln q = \ln a + \alpha \ln k + \beta \ln l$$

Other nonlinear functions can not be converted, for example the constant elasticity of substitution production function,

$$q = a\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]^{-\frac{1}{\beta}}$$

which becomes just another nonlinear function,

$$\ln q = \ln a - \frac{1}{\beta}\ln\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]$$

others

Example's 17 and 18 give some examples of the use of nonlinear function in economics, namely the nonlinear total revenue and the nonlinear total cost.

Example 17. Let the total revenue be $r_t = pq$, and the demand function $p = a - bq$, where q is the quantity sold. Then r_t expressed as a function of q is nonlinear, for $r_t = (a - bq)q = aq - bq^2$.

Example 18. A more realistic equation for the total cost instead of $c_t = a + bq$ is the nonlinear function $c_t = aq^3 - bq^2 + cq + d$ in which the production cost increases with quantity in at a decreasing rate ($c_t' < 0$) up to the inflection point at $q = \frac{b}{6a}$, after which it increases at an increasing rate ($c_t'' > 0$). During the first stage the cost per unit decreases once the initial investment has been spent. During the second stage the cost per unit increases since more capital needs to be invested in order to allow more production capacity.

Definition 11 talks about polynomial, and Example 19 about quadratic equation.

Definition 11. A *polynomial* is an expression in the form $\sum_{i=0}^{n} a_i x^{n-i}$. Here n is called the *order* of the polynomial. If $n = 2$ the polynomial is known as a *quadratic polynomial*, if $n = 3$ a *cubic polynomial*, if $n = 4$ a *quartic*, $n = 5$ a *quintic* and $n = 6$ a *sextic*. If we let $\mathrm{p}(x)$ be a polynomial, then a *polynomial equation* is the equation $\mathrm{p}(x) = 0$. A *polynomial function* is a function of the form $y = \mathrm{f}(x) = \mathrm{p}(x)$.

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Example 19. The quadratic equation $ax^2 + bx + c = 0$ has the solutions,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

These solutions are called the *roots* of the quadratic equation. Equation 3 is called the 'minus-b formula'. The values of x obtained from the minus-b formula give the intersections of the graph of the quadratic function

$$\mathrm{f}(x) = \mathrm{p}(x) = ax^2 + bx + c$$

on the x-axis. The value $b^2 - 4ac$ determines how the graph of $\mathrm{f}(x)$ lies relative to the x-axis, that is,

$$b^2 - 4ac \begin{cases} > 0, & \text{there are two } x\text{-intersections} \\ = 0, & \text{the graph touches the } x\text{-axis at one point} \\ < 0, & \text{the graph never touches the } x\text{-axis} \end{cases}$$

Furthermore, the graph reverses its direction with respect to the y-axis at the critical point where $\mathrm{f}'(x) = 0$, that is when $x = -\frac{b}{2a}$. Consequently the critical point is

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$$

The graph of $\mathrm{f}(x)$ is symmetric with respect to the vertical line which passes through the turning point, that is to say, the line $x = -\frac{b}{2a}$ The y-intercept is at the point $(0, c)$.

Example 20 is about another form of nonlinear function, the hyperbolic function.

Example 20. A hyperbolic relation is an expression of the form,

$$(px+q)(ry+s)=t$$

From this we obtain,

$$\begin{aligned}\left(x+\frac{q}{p}\right)\left(y+\frac{s}{r}\right) &= \frac{t}{pr}\\ y &= \frac{t}{pr}\left(\frac{1}{x+\frac{q}{p}}\right)-\frac{s}{r}\\ &= \frac{a}{x+b}-c\end{aligned}$$

where p, q, r, s and t are constants, hence so are $a=\frac{t}{pr}$, $b=\frac{q}{p}$, $c=\frac{s}{r}$. In economics we sometimes find hyperbolic functions of the form,

$$y=\frac{a}{bx+c} \tag{4}$$

For example, a demand function of a good may be given by,

$$q+a=\frac{m}{p}$$

which leads to,

$$p=\frac{m}{q+a}$$

where p and q are respectively price and quantity demanded of a good, while m and a are constants.

The graph of Equation 4 has the x-axis, that is the line $y=0$, as its horizontal asymptote, and has the line $x=-c/b$ as its vertical asymptote. If all the parameters are positive, then the curve in the first quadrant decreases with a decreasing rate.

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